THREE-DIMENSIONAL CAUCHY-POISSON PROBLEM FOR WAVES IN A VISCOUS FLUID

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PMM Vol.29, № 6, 1965, pp.1138-1146

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(Received February 10, 1965)

The three-dimensional problem of waves on the surface of a viscous fluid, caused by initial disturbances, is considered. Sretenskii considered an analogous problem for the plane case in [1], and for the three-dimensional case in [2]. In contrast to [2], wherein approximate formulas are given for the shape of the free surface for initial disturbances of just the delta-function type, without estimates of the degree of accuracy of the obtained approximation, approximate asymptotic formulas for arbitrary initial disturbances are obtained herein and an estimate of the degree of accuracy of the obtained approximation is given.

Moreover, in the case of concentrated initial disturbances of the delta-function type, formulas analogous to those of Kochin [3] for the solution of the analogous problem for an ideal fluid are obtained for the shape of the free surface.

1. At the initial time $t = 0$ let a viscous incompressible fluid occupying the half-space $z < 0$ be at rest, and let its free surface have the form $\zeta(x, y, 0) = \zeta_0(x, y)$. Let us pose the problem of finding the form of the free fluid surface at any time $t > 0$. Assuming the motions to be slow, we obtain the system of hydrodynamic equations in the form

$$\rho u_t = -p_x + \mu\Delta u, \quad \rho v_t = -p_y + \mu\Delta v, \quad \rho w_t = -p_z - \rho g + \mu\Delta w$$
$$u_x + v_y + w_z = 0 \tag{1.1}$$

with the initial conditions

$$u = v = w = 0, \quad \zeta = \zeta_0(x, y) \quad \text{for } t = 0 \tag{1.2}$$

and the boundary conditions

$$p_{nn} = p_{n\tau_1} = p_{n\tau_2} = 0, \quad \zeta_t = w \quad \text{for } z = \zeta \tag{1.3}$$

Here τ_1 and τ_2 are two orthogonal directions. Let us introduce the function $\psi(x, y, z, t)$ by means of the equality

$$p = \rho\psi - g\rho z \tag{1.4}$$

Then, since the motions are slight, to first order of accuracy inclusively, the boundary conditions (1.3) may be written as follows:

$$g\zeta = \psi - 2\nu w_z, \ \zeta_t = w, \quad u_z + w_x = 0, \quad v_z + w_y = 0 \quad \text{for } z = 0 \tag{1.5}$$

Taking account of (1.4), the system of equations (1.1) then becomes

$$u_t = -\psi_x + \nu \Delta u, \quad v_t = -\psi_y + \nu \Delta v, \quad w_t = -\psi_z + \nu \Delta w, \quad u_y + v_y + w_z = 0 \quad (1.6)$$

Let us apply the Laplace transform in the time t and the Fourier transform in the variables x and y to (1.6) and the boundary conditions (1.5). Then, taking account of the initial conditions (1.2), we obtain the following system of equations (1.7)

$$\alpha U = -im\Psi + \nu[-(m^2+n^2)U + U_{zz}], \quad \alpha V = -in\Psi + \nu[-(m^2+n^2)V + V_{zz}]$$

$$\alpha W = -\Psi_z + \nu[-(m^2+n^2)W + W_{zz}], \qquad i(mU + nV) + W_z = 0$$

with boundary conditions

$$\alpha[\Psi - 2\nu W_z - gZ_0] - gW = 0, \quad imW + U_z = 0, \quad inW + V_z = 0 \text{ for } z = 0 \quad (1.8)$$

Here U, V, W, Ψ, Z_0 are the transforms of the functions u, v, w, ψ, ζ_0 are functions of z, α, m, n, where α is the Laplace transform parameter, and m and n are Fourier transform parameters in x and y, respectively. From Equations (1.7) we find

$$\Psi_{zz} = (m^2 + n^2)\Psi$$

Hence, by satisfying the condition $\psi \to 0$ as $z \to -\infty$ we have

$$\Psi = A \exp(z\sqrt{m^2+n^2}) \quad (1.9)$$

Here that value is taken of the square root for which the real part is positive.

Taking account of (1.9) and the condition $U, V, W \to 0$ as $z \to -\infty$, we obtain from the system (1.7)

$$U = B\exp(zb\nu^{-1/2}) + B_1\exp(zr), \qquad B_1 = -im\alpha^{-1}A \qquad b = \sqrt{\alpha + \nu r^2}$$

$$V = C\exp(zb\nu^{-1/2}) + C_1\exp(zr), \qquad C_1 = -in\alpha^{-1}A, \qquad C = (ibD - mB)n^{-1} \quad (1.10)$$

$$W = D\exp(zb\nu^{-1/2}) + D_1\exp(zr), \qquad D_1 = -r\alpha^{-1}A, \qquad r = \sqrt{m^2+n^2}$$

Here A, B, D are arbitrary functions of α, m, n. Satisfying the boundary conditions (1.8), we obtain the following system to determine A, B, D

$$A(\alpha^2 + 2\nu\alpha r^2 + gr) - D\alpha(2\nu^{1/2}b\alpha + g) = \alpha^2 g Z_0$$

$$-2imr\nu^{1/2}A + bim\alpha\nu^{1/2}D + \alpha bB = 0 \quad (1.11)$$

$$-2in^2r\nu A + i\nu\alpha(b^2 + n^2\nu)D - m\alpha\nu^{1/2}bB = 0$$

Applying the integral transformation to the second equation of (1.5), we have $\alpha(Z - Z_0) = W$ for $z = 0$. Hence, we find from (1.10)

$$Z = Z_0 + \alpha^{-1}(D - A\alpha^{-1}r) \quad (1.12)$$

Substituting A and D the solutions of (1.11), here, we obtain

$$Z = Z_0(1 - gr\Delta_1^{-1}), \qquad \Delta_1 = \alpha^2 + gr + 4\nu\alpha r^2 - 4\nu^{3/2}br^3 + 2\nu^2 r^4$$

Introducing the variables $m = kr\cos\theta$, $n = kr\sin\theta$, $\alpha = \sigma\alpha_1$, $x = R\cos\gamma$, $y = R\sin\gamma$ and applying the Mellin formula and the inverse Fourier transform, we obtain from (1.12)

$$\zeta(x,y,t) = \frac{\sigma^4}{2\pi g^2}[\eta_1 + \varepsilon^{1/2}(\eta_2 + \eta_3)] \quad (1.13)$$

$$\eta_j = \int_0^{2\pi}\int_0^{\infty} Z_0(r,\theta)\varkappa_j(r,t,\varepsilon)\exp(irR_1\cos(\theta-\gamma))\, r\,dr\,d\theta, \quad \varkappa_j = \frac{1}{2\pi i}\int_{s-i\infty}^{s+i\infty}\varphi_j(\alpha, r)\, e^{\alpha t} d\alpha$$

$$Z_0 = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty}\zeta_0(R,\gamma)\exp(-irR_1\cos(\theta-\gamma))\, R\,dR\,d\gamma \qquad (j = 1, 2, 3)$$

$$\varphi_1 = \frac{\alpha + 4\varepsilon r^2}{\Delta}, \quad \varphi_2 = \frac{-4r^3 \sqrt{\alpha + \varepsilon r^2} + 2\varepsilon^{1/2} r^4}{\alpha \Delta_1}, \quad \varphi_3 = \frac{(\alpha + 4\varepsilon r^2)\Delta_2}{\Delta \Delta_1} \qquad \text{(1.13) cont.}$$

$$\Delta = \alpha^2 + 4\varepsilon\alpha r^2 + r + 4\varepsilon^2 r^4, \quad \Delta_1 = \Delta - \varepsilon^{3/2}\Delta_2, \quad \Delta_2 = 4r^3\sqrt{\alpha + \varepsilon r^2} + 2\varepsilon^{1/2} r^4$$

$$\varepsilon = \nu g^{-2}\sigma^3, \quad k = \sigma^2 g^{-1}, \quad R_1 = kR, \quad \sigma = 1 \text{ сек}^{-1}$$

The expression (1.13) is the exact solution of the formulated problem for the arbitrary function $\zeta_0(x,y)$, relative to which we assume that it admits of a Fourier transformation in the variables x and y. Hence, $Z_0(r, \theta)$ has no singularities in the domain of integration and for $r \to +\infty$ tends to zero more rapidly than r^{-1}. Let us analyze (1.13) further by assuming $\varepsilon \ll 1$. It can be shown that by neglecting quantities of order $\varepsilon^{1/2}$ and higher, (1.13) may be written as follows:

$$\zeta = \frac{\sigma^4}{2\pi g^2}\eta_1 \tag{1.14}$$

For this it is evidently necessary to show that η_2 and η_3 have a finite limit as $\varepsilon \to 0$. Using the Vallée-Poussin criterion [4] on uniform convergence of integrals with infinite limits, it is easy to show that the integrals η_2 and η_3 converge uniformly in respect to ε, and then we can pass to the limit under the integral sign. Let us evaluate the expression $(\varkappa_3)_{\varepsilon=0} = \varkappa_3^\circ$, which has the form

$$\frac{2r^3}{\pi i}\int_{s-i\infty}^{s+i\infty} \frac{\alpha\sqrt{\alpha}}{(\alpha^2+r)^2} e^{\alpha t} d\alpha.$$

By contour integration we find

$$\varkappa_3^0 = -2\,\mathrm{Re}\left[i^{3/2}\left(t - \frac{i}{2\sqrt{r}}\right)\right] r^{11/4} + K_3, \quad K_3 = \frac{4r^3}{\pi}\int_0^\infty \frac{\alpha^{3/2}}{(\alpha^2+r)^2} e^{-\alpha t}\, d\alpha \tag{1.15}$$

The last integral is expressed in terms of Mayer functions and has the form

$$K_3 = \frac{2r^{9/4}}{\pi^{3/2}} G_{13}^{31}\left(\frac{rt^2}{4}\,\middle|\,{}^{-1/4}_{3/4,\,0,\,1/2}\right)$$

Since any pair of the numbers $b_1 = 3/4,\ b_2 = 0,\ b_3 = 1/2$ does not differ by an integer, then the following representation will be valid:

$$G_{13}^{31}(\xi\,|\,{}^{a_1}_{b_1, b_2, b_3}) = \sum_{h=1}^{3}\prod_{j=1}^{3}{}' \,\Gamma(b_j - b_h)\,\Gamma(1 + b_h - a_1)\,\xi^{b_h} \times$$
$$\times F_2^{(1)}(1 + b_h - a_1;\ \ 1 + b_h - b_\alpha,\ \ 1 + b_h - d_\gamma;\ \xi)$$

where α and γ take the values 1, 2, 3 upon compliance with the conditions $\alpha \neq h, \gamma \neq h, \alpha < \gamma, a_1 = -1/4$, $\xi = 1/4\, rt^2$. Hence, it is seen that for $r \to 0$, $K_3 \to 0$ as $r^{1/4}$. For $t > 0$ and $r \to \infty$ it is easy to show that $K_3 \to 0$ no more rapidly than r^1, and therefore for $r \to \infty$, $\varkappa_3^\circ \to \infty$ as $r^{11/4}$. On the basis of (1.13) and (1.15) we have the following representation

$$\eta_3^0 = \sum_{k=1}^{5} B_k \tag{1.16}$$

$$B_{1,2} = t\int_0^{2\pi}\int_0^{\infty} \varphi_{1,2}(r,\theta)\exp\,[iRr\cos(\theta - \gamma] \pm i\sqrt{r}t]\,dr\,d\theta, \quad \varphi_{1,2} = -(\pm i)^{3/2} r^{15/4} Z_0(r,\theta)$$

$$B_{3,4} = \int_0^{2\pi}\int_0^{\infty} \varphi_{3,4}(r,\theta)\exp\,[iRr\cos(\theta - \gamma) \pm i\sqrt{r}t]\,dr\,d\theta, \quad \varphi_{3,4} = -1/2\,(\pm i)^{1/2} r^{13/4} Z_0(r,\theta)$$

$$B_5 = \int_0^{2\pi}\int_0^{\infty} \varphi_5(r,\theta)\exp\,[iRr\cos(\theta - \gamma)]\,dr\,d\theta, \quad \varphi_5 = K_3 Z_0(r,\theta)$$

Let us assume that for $r \to \infty$, $Z_0(r, \theta) \to 0$ not more slowly than r^{-5} for appropriate constraints on $\zeta_0(x, y)$. Under this condition all the integrals B_k will converge absolutely. The expression $B_{1,2}$ contains the factor t in front of the double integral. This integral may be easily shown to be a quantity on the order of t^{-1} for large t by the method of stationary phase. Hence, as t grows $B_{1,2}$ will remain bounded. Thus η_3 remains bounded as $\varepsilon \to 0$. It is shown completely analogously that η_2 will remain bounded as $\varepsilon \to 0$ for the same assumptions on the function $Z_0(r, \theta)$. Therefore, (1.14) yields an expression for the shape of the free fluid surface to first degree of accuracy in ε, inclusive.

On the basis of the Jordan lemma the integral $\varkappa_1$ is evaluated by using residue theorems. Since the equation $\Delta = 0$ has the roots $\alpha_{1,2} = -2\varepsilon r^2 \pm i\sqrt{r}$, then

$$\varkappa_1 = -\sum_{k=1}^{2} \frac{\alpha_k + 4\varepsilon r^2}{\alpha_1 - \alpha_2} e^{\alpha_k t} (-1)^k \tag{1.17}$$

Now (1.14) takes the form

$$\zeta = \frac{\sigma^4}{2\pi g^2} \int_0^{2\pi} \int_0^{\infty} Z_0(r, \theta) \varkappa_1(r, t) \exp[iRr\cos(\theta - \gamma)]\, r\, dr\, d\theta \tag{1.18}$$

Here $\varkappa_1(r, t)$ is given by (1.17). It is easy to verify that the initial condition $\zeta(x, y, 0) = \zeta_0(x, y)$ is satisfied.

The first degree of accuracy in ε inclusive, the obtained Formula (1.18) represents the shape of the free surface of the fluid at any time $t > 0$.

2. At the initial time $t = 0$ let the free surface of the fluid have the form

$$\zeta_0(x, y) = \begin{cases} \zeta_0(1 - R^2a^{-2})^{\mu} & (R < a) \\ 0 & (R > a) \end{cases} \quad \begin{pmatrix} \mu > 0 \\ \zeta_0 = \text{const} \end{pmatrix} \tag{2.1}$$

In this case

$$Z_0(r, \theta) = \zeta_0 a^2 k^{-2} \varphi(r), \qquad \varphi(r) = 2^{\mu}\Gamma(\mu + 1)(ar)^{-(\mu+1)} J_{\mu+1}(ar) \tag{2.2}$$

and the integrals η_2 and η_3 will converge absolutely for $\mu > 3.25$. Taking into account that the integral in (1.18) is the limit of the integral

$$\int_0^{2\pi} \int_0^{\infty} Z_0(r, \theta) \varkappa_1(r, t) \exp[iRr\cos(\theta - \gamma) + zr]\, r dr\, d\theta$$

as $z \to -0$, let us write the expression for ζ as

$$\zeta = \zeta_0 \frac{\sigma^4 a^2}{g^2 k^2} \int_0^{\infty} \varphi(r) \varkappa_1(r, t) J_0(rR) e^{zr} r\, dr \tag{2.3}$$

Let us note that the volume Q of the initial rise in the fluid is evaluated by means of Formula $Q = \pi a^2 \zeta_0 (\mu + 1)^{-1}$. To simplify the subsequent calculations let us consider the limiting case when $a \to 0$, $\zeta_0 \to \infty$ so that the quantity Q remains invariant (an initial rise of delta-function type). In this limiting case $\varphi(r)$ equals $[2(\mu + 1)]^{-1}$ and (2.3) takes the form

$$\zeta = \frac{1}{2\pi} Q\sigma^4 g^{-2} \lim_{z \to -0} \psi(\varepsilon), \qquad \psi(\varepsilon) = \int_0^{\infty} rJ_0(rR) \varkappa_1(r, t) e^{zr} dr$$

$$\varkappa_1 = (\cos\sqrt{r}t + 2\varepsilon r^{3/2} \sin\sqrt{r}t) \exp(-2\varepsilon r^2 t) \tag{2.4}$$

Expanding the function $\psi(\varepsilon)$ in a series in the parameter ε and being limited to the first three terms of the series, we have

$$\psi(\varepsilon) = \psi(0) + \varepsilon\psi'(0) + \varepsilon^2\, {}^1\!/_2\varphi''(0) + R_3$$

$$\psi^{(k)}(0) = 2^{k+1}(-1)^k r^{2k} \int_0^{\infty} [rt^k \cos\sqrt{r}t - kt^{k-1}\sqrt{r}\sin\sqrt{r}t] J_0(rR) e^{zr} dr \qquad (k = 0, 1, 2)$$

Using the expression for the remainder term of the Taylor series [4], we find that R_3 will be on the order of ε^3. Expanding $\sin\sqrt{rt}$ and $\cos\sqrt{rt}$ in series, we obtain (2.4) as

$$\zeta = \frac{Q\sigma^4}{2\pi g^2}\sum_{k=0}^{\infty}(-1)^k(\alpha_k + 2\varepsilon\beta_k + 4\varepsilon^2\gamma_k), \qquad \left(A_k = \int_0^{\infty} r^k J_0(rR)\, e^{zr}\, dr\right)$$

$$\alpha_k = \frac{1}{(2k)!}t^{2k}A_{k+1}, \qquad \beta_k = -\frac{2k}{(2k+1)!}t^{2k+1}A_{k+3}, \qquad \gamma_k = \frac{2k-1}{(2k+1)!}t^{2k+2}A_{k+5}$$

Taking into account that

$$A_{k+1} = (z^2+R^2)^{-1/2(k+2)}\Gamma(k+2)\,P^0_{k+1}\left(-z(z^2+R^2)^{-1/2}\right)$$

$$P^0_{k+1}(0) = \sqrt{\pi}\,[\Gamma(k+{}^3/_2)\,\Gamma(-{}^1/_2 k)]^{-1} \tag{2.5}$$

where Γ is the gamma function, P_{k+1} spherical functions of the first kind, we find a final expression for the rise ζ

$$\zeta = \frac{Q\omega}{2\pi}R^{-2}\left[H_0 + \frac{\nu t}{R^2}H_1 + \left(\frac{\nu t}{R^2}\right)^2 H_2\right], \quad \omega = \frac{gt^2}{2R}$$

$$H_j = \sum_{k=0}^{\infty}(-1)^k\delta_{kj}\omega^{2k} \quad (j = 0, 1, 2), \qquad \delta_{k0} = \frac{[(2k+1)!!]^2\, 2^{2k+1}}{[2(2k+1)]!}$$

$$\delta_{k1} = \frac{2(4k+2)\,[(2k+3)!!]^2\, 2^{2k+1}}{(4k+3)!}, \qquad \delta_{k2} = \frac{4(4k+1)\,[(2k+5)!!]^2\, 2^{2k+1}}{(4k+3)!}$$

Here t and R are the initial dimensional quantities. For $\nu = 0$, Expression (2.6) agrees with the expression for the rise in the fluid obtained in solving the analogous problem for an ideal fluid [3 and 5].

The series (2.6) for the H_j will converge for any values of ω, but they are convenient for calculations only for small values of ω. Kochin [3] found an expression for the series H_0 in terms of Bessel functions which is convenient for calculations for any values of ω. This expression is

$$H_0 = {}^1/_8\pi\sqrt{2}J_{1/4}J_{-1/4} - {}^1/_{32}\pi\sqrt{2}\omega\,(J_{1/4}J_{3/4} - J_{-1/4}J_{-3/4}) \tag{2.7}$$

All these functions have the same argument $\frac{1}{4}\omega$. Let us now find an expression for the series H_1 in terms of Bessel functions. This series may be represented by an integral of the form

$$H_1 = \int_0^{1/2\pi} F({}^1/_2\omega\cos^2\varphi)\cos^3\varphi\, d\varphi \quad (F(x) = 2\,[9 - 6x^2)\,J_0(x) - (10x - x^3)\,J_1(x)]) \tag{2.8}$$

It is easy to verify (2.8) by substituting the series expressions for J_0 and J_1 and then integrating term by term. On the other hand, by using Formula

$$J_\mu(z)\,J_\nu(z) = \frac{2}{\pi}\int_0^{1/2\pi} J_{\mu+\nu}(2z\cos\theta)\cos(\mu-\nu)\,\theta\, d\theta \tag{2.9}$$

known from Bessel-function theory, the integral (2.8) may be expressed, if $\mu + \nu > -1$, in terms of Bessel functions as

$$H_1 = {}^1/_2\pi\sqrt{2}\,[{}^9/_4J_{1/2}J_{-1/2} + {}^{21}/_8J_1J_0 - {}^{295}/_{64}J_{3/4}J_{-3/4} + {}^5/_{32}J_{-1/4}J_{-3/4} -$$
$$- {}^{33}/_{16}J_{1/4}J_{-1/4} + {}^1/_{32}\omega\,(53J_{-1/4}J_{-3/4} - 58J_{1/4}J_{3/4}) -$$
$$- {}^1/_{64}\omega^2\,(36J_{1/4}J_{-1/4} + 33J_{3/4}J_{-3/4}) + {}^{17}/_{16}\omega^3\,(J_{1/4}J_{3/4} - J_{-1/4}J_{-3/4})]$$

Here all the functions have the same argument $\frac{1}{4}\omega$. Similarly, an expression in terms of Bessel functions may be obtained for H_2 by representing it as

$$H_2 = 4V_1 - 8t\,\frac{dV_2}{dt}$$

$$V_1 = \int_0^{1/2\pi} F_1(1/2\omega\cos^2\varphi)\cos\varphi\,d\varphi, \qquad V_2 = \omega\int_0^{1/2\pi} F_1(1/2\omega\cos^2\varphi)\cos^3\varphi\,d\varphi$$

$$F_1(x) = (225 + 263x^2 + 15x^4)J_0(x) - (x^5 - 70x^3 + 252x)J_1(x)$$

and integrating the expressions V_1 and V_2, just as (2.8).

3. Let us conduct an asymptotic analysis of (1.18) for large values of R. For this let us write this expression as

$$\zeta = \frac{\sigma^4}{2\pi g^2}\int_0^{2\pi}\int_0^{\infty}\sum_{k=1}^{2} f_k(r,\theta)\,e^{iRM_k(r,\theta)}\,r\,dr\,d\theta \tag{3.1}$$

$$f_k = (-1)^k\,\frac{i\sqrt{r}\,(-1)^k - 2\varepsilon r^2}{2i\sqrt{r}}\,e^{-2\varepsilon r^2 t}Z_0(r,\theta)$$

$$M_k = r\cos(\theta-\gamma) - \xi\sqrt{r}\,(-1)^k, \qquad \xi = tR^{-1}$$

Using the known formula of the method of stationary phase for double integrals [6], we obtain the following expression for the rise in fluid for large R for an arbitray initial rise:

$$\zeta = \frac{2\sqrt{2}\sigma^2}{gR}\,\mathrm{Re}\left[f(r_1,\theta_1)\exp\left(i\frac{gt^2}{4R}\right)\right]\exp\left(-\frac{\nu g^2t^5}{8R^4}\right) \tag{3.2}$$

$$f(r,\theta) = \frac{i\sqrt{r} + 2\varepsilon r^2}{2i}\sqrt{r}Z_0(r,\theta), \qquad r_1 = \frac{\xi^2}{4}, \qquad \theta_1 = \pi + \gamma$$

Since the parameter ξ enters into Expression $M_k(r,\theta)$, it should be small as compared with R, according to which the asymptotic estimate of the integral (3.1) will be made. Hence, the asymptotic formula (3.2) will be valid for $0 \leqslant t \leqslant T$, where T is a quantity of the order of $g^{-1}R\sigma$. For an initial rise of the form (2.1) concentrated in the neighborhood of the origin and having the volume Q, Expression (3.2) becomes

$$\zeta = \frac{gt^2Q}{4\sqrt{2}\pi R^3}\left(\cos\frac{gt^2}{4R} + \frac{\nu gt^3}{4R^3}\sin\frac{gt^2}{4R}\right) \tag{3.3}$$

Let us note that the limiting value of (3.3) agrees, as $\nu\to 0$, with the expression for the wave shape which is the solution of the analogous problem for an ideal fluid [5].

To obtain the asymptotic estimate of (3.1) for large values of t, let us write (3.1) as

$$\zeta = \frac{\sigma^4}{2\pi g^2}\int_0^{2\pi}\int_0^{\infty}\sum_{k=1}^{2} f_k(r,\theta)\,e^{itN_k(r,\theta)}\,r\,dr\,d\theta$$

$$N_k(r,\theta) = \xi_1 r\cos(\theta-\gamma) - \sqrt{r}\,(-1)^k, \qquad \xi_1 = Rt^{-1}$$

and, just as before, we obtain that (3.2) will be valid even for large values of t for $0 \leqslant R \leqslant R_1$, where R_1 is a quantity of the order of t.

4. The problem of waves caused by an initial disturbance of the free surface was considered in the previous sections. Let us now consider the problem of waves caused by an arbitrary pressure pulse applied to a free surface.

During a very small time interval τ let an arbitrary pressure pulse be applied to the horizontal surface of a viscous fluid at rest, occupying the half-space $z < 0$. Then the projections of the velocities caused by this pressure pulse are defined by Formulas

$$u_0 = -\rho^{-1}\Phi_x, \quad v_0 = -\rho^{-1}\Phi_y, \quad w_0 = -\rho^{-1}\Phi_z \left(\Phi = \int_0^\tau p\, d\tau\right) \tag{4.1}$$

Since the pressure pulse is given only on the free surface, then the function $\Phi(x,y,z)$ will indeed be known only for $z = 0$. Let $\Phi(x,y,0) = F(x,y)$. Since the fluid is incompressible, then $\Delta\Phi = 0$, and hence the function

$$\Phi = \frac{1}{2\pi} \iint_{-\infty}^{\infty} K(m, n) \exp\left[i(mx + ny) + z\sqrt{m^2 + n^2}\right] dm\, dn$$

$$\left(K(m, n) = \frac{1}{2\pi} \iint_{-\infty}^{\infty} F(x, y) \exp\left[-i(mx + ny)\right] dx\, dy\right)$$

satisfying the Laplace equation and the required condition at $z = 0$, will be the desired function. Therefore, the initial velocities caused by the pressure pulse $F(x,y)$ are now known in the whole fluid. Since the time τ is infinitesimal, and the velocities are finite, we then neglect the displacement of the particles in the time τ. Taking the instant the pulse ceases to act as the initial time, we arrive at the following problem: to find the solution of the system (1.1) with the initial conditions $u = u_0$, $v = v_0$, $w = w_0$, $\zeta = 0$ at $t = 0$, where u_0, v_0, w_0 are given by (4.1) and the boundary conditions by (1.3). Applying the Laplace transform in the time t and the Fourier transform in the variables x and y to (1.6) and the boundary conditions (1.5), we obtain a system for the transforms if we take account of the initial conditions

$$U_{zz} - (r^2 + \alpha\nu^{-1})U - im\nu^{-1}\Psi = -\alpha\nu^{-1}U_0$$

$$V_{zz} - (r^2 + \alpha\nu^{-1})V - in\nu^{-1}\Psi = -\alpha\nu^{-1}V_0 \tag{4.2}$$

$$W_{zz} - (r^2 + \alpha\nu^{-1})W - \nu^{-1}\Psi_z = -\alpha\nu^{-1}W_0, \qquad i(mU + nV) + W_z = 0$$

Here the capital letters denote the transforms of the functions denoted by the lower case letters above, for this

$$U_0 = -imL, \quad V_0 = -inL, \quad W_0 = -rL, \quad L = \rho^{-1}K(m, n)e^{rz}$$

the boundary conditions (1.5) in the transforms (for $z = 0$) become

$$gZ = \Psi - 2\nu W_z, \quad \alpha Z = W, \quad U_z + imW = 0, \quad V_z + inW = 0 \tag{4.3}$$

Solving the system (4.2) with the boundary conditions (4.3) and then using the inversion formula, we find the expression for the rise in the fluid

$$\zeta = -\frac{\sigma^5}{2\pi\rho g^3}\left[\eta_1 + \varepsilon^{3/2}\eta_2 + \varepsilon^2\eta_3\right], \qquad \eta_j = \int_0^{2\pi}\int_0^{\infty} K(r, \theta)\, \varkappa_j(r, t, \varepsilon)\, e^{iRr\cos(\theta-\gamma)} r\, dr\, d\theta \tag{4.4}$$

$$\varkappa_j = \frac{1}{2\pi i}\int_{s-i\infty}^{s+i\infty} \varphi_j(\alpha, r)\, e^{\alpha t}\, d\alpha \quad (j = 1, 2, 3), \quad \varphi_1 = \frac{r}{\Delta}, \quad \varphi_2 = \frac{\Delta_2}{\Delta\Delta_1}, \quad \varphi_3 = -\frac{2r^4}{\Delta_1}$$

The quantities $\Delta, \Delta_1, \Delta_2, \varepsilon, k$ are given by (1.13). Expression (4.4) is the exact solution of the formulated problem for the arbitrary function $F(x,y)$ relative to which we shall assume that it admits of a Fourier transform in the variables x and y. By the same method as in Section 1, it may be shown that the expressions for η_2 and η_3 have a finite limit as $\varepsilon \to 0$ under the same constraints on the function $F(x,y)$ as for $\zeta_0(x,y)$ Hence, for $\varepsilon \ll 1$, Expression (4.4) may be written with an error of the order of $\varepsilon^{3/2}$ as

$$\zeta = -\frac{\sigma^5}{2\pi\rho g^3}\int_0^{2\pi}\int_0^{\infty} r^{3/2}K(r, \theta)\sin\sqrt{r}t \exp\left[iRr\cos(\theta - \gamma) - 2\varepsilon r^2 t\right] dr\, d\theta \tag{4.5}$$

Thus, to first degree of accuracy in ε inclusive, Formula (4.5) yields the shape of waves resulting from the pressure pulse under consideration.

5. Let the function $F(x, y)$ be

$$F(x, y) = \begin{cases} \Pi(1 - R^2 a^{-2})^{\mu} & (R \leqslant a) \\ 0 & (R > a) \end{cases} \qquad \begin{pmatrix} \mu > 0 \\ \Pi = \text{const} \end{pmatrix} \tag{5.1}$$

In this case $K(r, \theta) = \Pi a^2 k^{-2} \varphi(r)$, where $\varphi(r)$ is given by (2.2) and the integrals η_2 and η_3 will converge absolutely for $\mu > 3.5$. Taking into account that the integral in (4.5) is the limit of the integral

$$\int_0^{2\pi}\int_0^{\infty} r^{3/2} K(r, \theta) \sin \sqrt{r} t \exp[iRr\cos(\theta - \gamma) - 2\varepsilon r^2 t + zr]\, dr\, d\theta$$

as $z \to -0$, let us write Expression for ζ as

$$\zeta = -\frac{\sigma^5 \Pi a^2}{\rho g^3 k^2} \int_0^{\infty} \varphi(r) J_0(rR) \sin \sqrt{r} t \exp[iRr\cos(\theta - \gamma) - 2\varepsilon r^2 t + zr]\, r^{3/2}\, dr\, d\theta \tag{5.2}$$

Let us note that the total magnitude of the pressure pulse q is calculated by means of Formula $q = \pi a^2 \Pi(\mu + 1)^{-1}$. To simplify the subsequent calculations let us consider the limiting case $a \to 0$, $\Pi \to \infty$ so that the quantity q remains invariant. In this limiting case $\varphi(r)$ equals $[2(\mu + 1)]^{-1}$, and Expression (5.2) becomes

$$\zeta = -\frac{q\sigma^5}{2\pi\rho g^3} \lim_{z \to -0} \psi(\varepsilon), \qquad \psi(\varepsilon) = \int_0^{\infty} r^{3/2} J_0(rR) \sin \sqrt{r} t e^{-2\varepsilon r^2 t + zr}\, dr$$

Expanding $\psi(\varepsilon)$ in a power series in ε and limiting ourselves to the first three terms of the series, we have

$$\psi(\varepsilon) = \psi(0) + \varepsilon\psi'(0) + {}^1\!/_2 \varepsilon^2 \psi''(0) + R_3$$

$$\psi^k(0) = (-2t)^k \int_0^{\infty} r^{3/2 + 2k} J_0(rR) \sin \sqrt{r} t e^{zr}\, dr$$

Here R_3 is a quantity of the order of ε^3 on the basis of the formula for the remainder term of the Taylor series. Expanding $\sin \sqrt{r}t$ in series and taking account of (2.5), we find the following expression for the rise ζ :

$$\zeta = \frac{qt}{2\pi\rho R^3}\left[H_0 + \frac{\nu t}{R^2} H_1 + \frac{\nu^2 t^2}{R^4} H_2\right], \qquad H_j = \sum_{k=0}^{\infty} (-1)^k \delta_{kj} \omega^{2k} \tag{5.3}$$

$$\omega = \frac{gt^2}{2R}, \qquad \delta_{k0} = \frac{[(2k+1)!!]^2 2^{2k}}{(4k+1)!}, \qquad \begin{matrix} \delta_{k1} = \delta_{k0}(2k+3)^2 \\ \delta_{k2} = \delta_{k1}(2k+5)^2 \end{matrix} \qquad (j = 1, 2, 3)$$

Here t and R are the initial dimensional variables. The expression for $\nu = 0$ agrees with the expression for the rise in fluid obtained in solving the analogous problem for an ideal fluid [3]. The obtained series will converge for any values of ω but they are convenient for calculations only for small ω. Kochin [3] found an expression for the series H_0 in terms of Bessel functions which is convenient for calculations for any ω This expression is

$$H_0 = \frac{\pi\sqrt{2}}{32}\left[2J_{1/4}J_{-1/4} - 3\omega(J_{3/4}J_{1/4} - J_{-3/4}J_{-1/4}) - \omega^2(J_{1/4}J_{-1/4} + J_{3/4}J_{-3/4})\right]$$

where all the functions have the same argument $\frac{1}{2}\omega$. Let us now find an expression for the series H_1 in terms of Bessel functions. This series may be represented by

$$H_1 = \int_0^{1/2\pi} F({}^1\!/_2 \omega \cos^2\varphi) \cos\varphi\, d\varphi \tag{5.4}$$

$$F(x) = 2\left[(9 - 17x^2 + x^4) J_0(x) + (-12x + 6x^3) J_1(x)\right]$$

Using (2.9), we find the following expression for the integral (5.4) in terms of Bessel functions:

$$H_1 = {}^1/_2 \sqrt{2\pi}\,[{}^1/_{64}\omega^4 (J_{3/4}J_{-3/4} + {}^1/_4 J_{1/4}J_{-1/4}) + {}^{21}/_{64}\omega^3 (J_{1/4}J_{3/4} - J_{-1/4}J_{-3/4}) -$$
$$- {}^1/_{64}\omega^2 (335 J_{1/4}J_{-1/4} + 741 J_{3/4}J_{-3/4}) + {}^{41}/_{64}\omega (J_{3/4}J_{1/4} - J_{-3/4}J_{-1/4}) -$$
$$- {}^3/_2\omega (J_0 J_1 - J_{-1/2}^2) + {}^{329}/_{32} J_{-1/4}J_{1/4} - 6 J_{1/2}J_{-1/2}]$$

By a similar method we may obtain an expression for the series H_2 in terms of Bessel functions by representing it as

$$H_2 = \int_0^{1/2\pi} F_1 ({}^1/_2\omega \cos^2\varphi) \cos\varphi \, d\varphi \qquad (5.5)$$

$$F_1(x) = 2\,[(-x^6 + 96x^4 - 568x^2 + 225)\,J_0(x) + (-15x^5 + 223x^3 + 195x)\,J_1(x)]$$

and then integrating (5.5) analogously to (5.4).

6. Let us carry out an asymptotic analysis of (4.5) for large R.

Let us write this expression as

$$\zeta = \frac{\sigma^5}{4\pi i \rho g^3} \int_0^{2\pi}\int_0^{\infty} \sum_{k=1}^{2} f_k(r,\theta) \exp\,[iRM_k(r,\theta)]\, dr\, d\theta \qquad (6.1)$$

$$f_k(r,\theta) = r^{3/4} K(r,\theta) \exp(-2\varepsilon r^2 t)(-1)^k$$

Here $M_k(r,\theta)$ is given by (3.1). For large R we obtain the following asymptotic expression for the rise in fluid due to an arbitrary initial pulse:

$$\zeta = -\frac{\sqrt{2}\sigma^3}{\rho g^2 R} \operatorname{Im}\left\{f(r_1,\theta_1) \exp\left(i\frac{gt^2}{4R}\right)\right\} \exp\left(-\frac{\nu g^2 t^5}{8R^4}\right) \qquad (6.2)$$

$$f(r,\theta) = r^{3/2} K(r,\theta), \qquad r_1 = {}^1/_4\xi^2, \qquad \theta_1 = \pi + \gamma.$$

Formula (6.2) will be valid for $0 \leqslant t \leqslant T$, where T is a quantity of the order of $g^{-1}R\sigma$. The quantities R and t in (6.2) are dimensional. For an initial pulse of the form (5.1) concentrated in the neighborhood of the origin, whose total value is q, Expression (6.2) becomes

$$\zeta = -\frac{qgt^3}{8\sqrt{2\pi}\rho R^4} \sin\frac{gt^2}{4R} \exp\left(-\frac{\nu g^2 t^5}{8R^4}\right) \qquad (6.3)$$

Exactly as has been shown in Section 3, we obtain that (6.2) will be valid for large values of t for $0 \leqslant R \leqslant R_1$, where R_1 is a quantity of the order of t.

The obtained Formulas (6.2) and (6.3) afford a possibility of constructing a pattern of the free surface motion by elementary means at some distance from the domain of application of the pressure pulse in the mentioned time range, and also an analogous pattern for large values of time t in the mentioned range of variation of R.

BIBLIOGRAPHY

1. Sretenskii, L.N., O volnakh na poverkhnosti viazkoi zhidkosti (On waves on a viscous fluid surface). Trudy tsent.aero-gidrodin.Inst.(TsAGI) № 541, 1941.
2. Nikitin, A.K. and Podrezov, S.A., K prostranstvennoi zadache o volnakh na poverkhnosti viazkoi zhidkosti beskonechnoi glubiny (On the spatial problem of waves on the surface of a viscous fluid of infinite depth). *PMM* Vol.28, № 3, 1964.
3. Kochin, N.E., Sobr.soch. (Collected Works), Vol.2, Izd.Akad.Nauk SSSR,1949.
4. Whittaker, E.T. and Watson, G.N., Kurs sovremennogo analiza (Modern Analysis). Part I, (Russian translation). Fizmatgiz, 1963.
5. Sretenskii, L.N., Teoriia volnovykh dvizhenii zhidkosti (Theory of the Wave Motions of Fluid). ONTI, 1936.
6. Fedoriuk, M.V., Metod statsionarnykh faz dlia mnogomernykh integralov (Statioary phase method for multidimensional integrals). Zh.vychisl. Mat.mat.Fiz., Vol.2, № 1, 1962.

Translated by M.D.F.